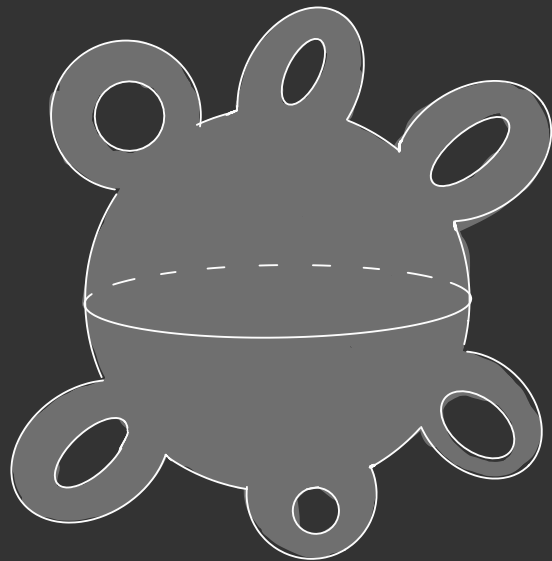
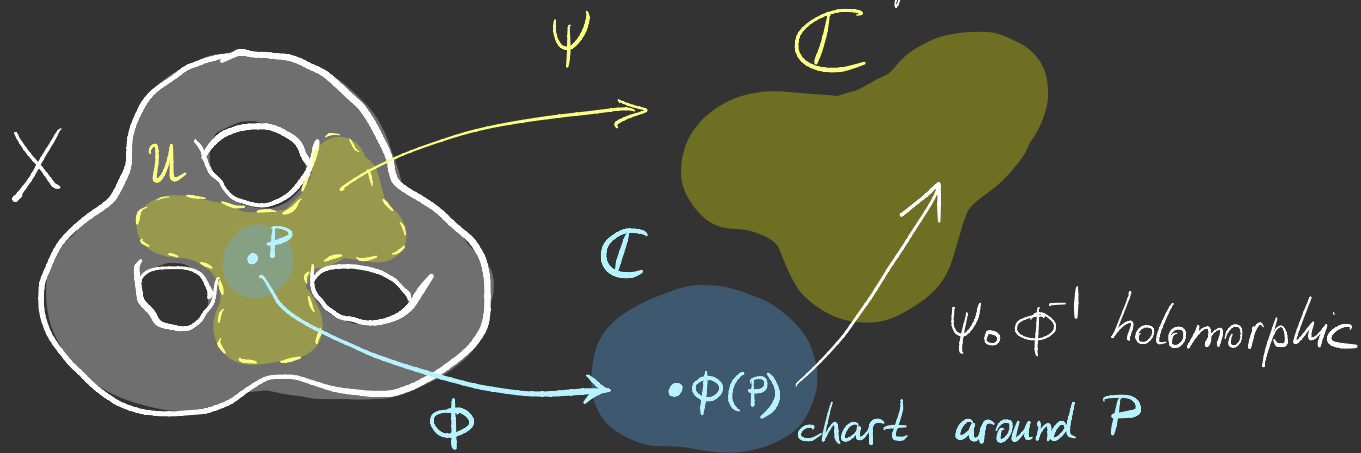


Genus of a modular
curve



Maps between Riemann surfaces

Definition: Let X be a Riemann surface. A complex valued function ψ defined on an open set $U \subset X$ is **holomorphic** if and only if for any $P \in U$ and any local coordinate ϕ at P $\psi \circ \phi^{-1}$ is holomorphic near $\phi(P)$.



Maps between Riemann surfaces

Definition: A continuous map $f: Y \rightarrow X$ between two Riemann surfaces is **holomorphic** if for any holomorphic chart $\Phi: \mathcal{U} \rightarrow \mathbb{C}$ in X the map $\Phi \circ f: f^{-1}(\mathcal{U}) \rightarrow \mathbb{C}$ is holomorphic.

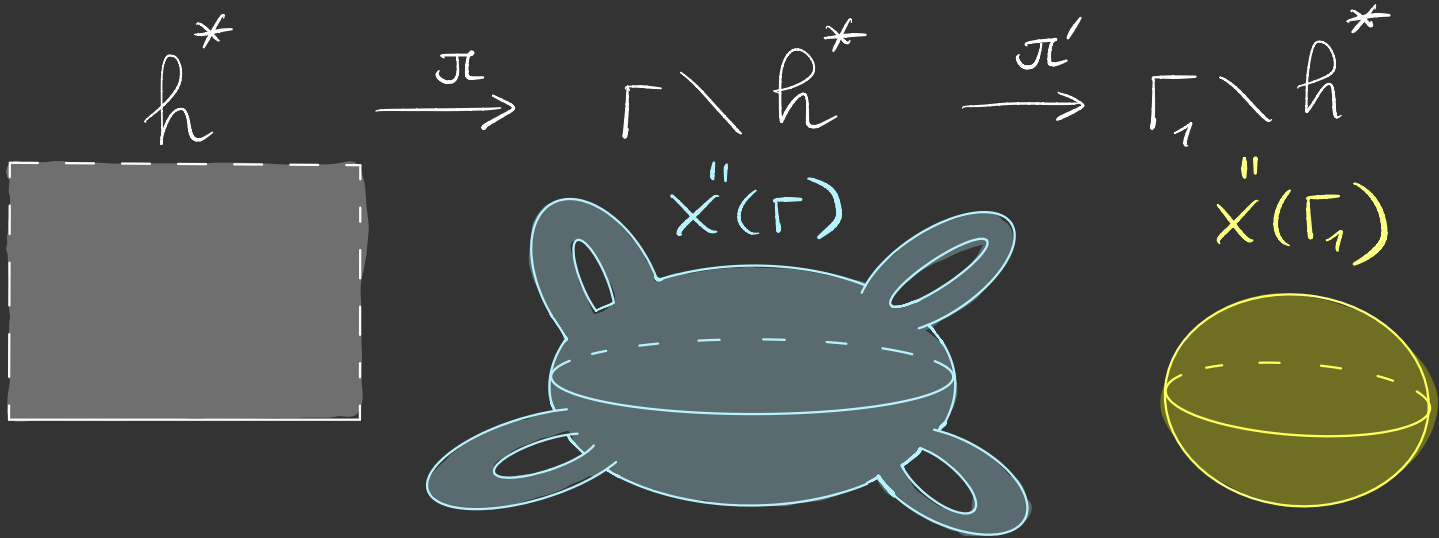
Such a map is called an **isomorphism of Riemann surfaces** (or **biholomorphic map**) if it is a homeomorphism and its inverse is holomorphic.

Example

$X(1)$ — compact Riemann surface $SL_2(\mathbb{Z}) \backslash \mathbb{H}^*$

Γ — congruence subgroup of $SL_2(\mathbb{Z})$

$X(\Gamma)$ — compact Riemann surface $\Gamma \backslash \mathbb{H}^*$



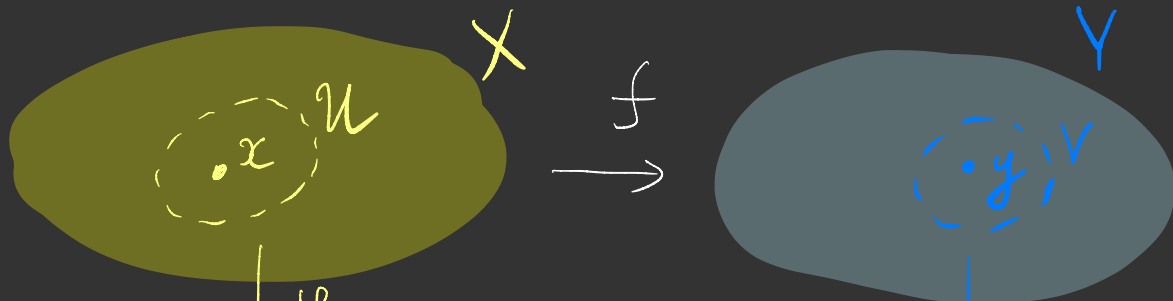
Multiplicity of a holomorphic map at a point

Proposition: Let $f: X \rightarrow Y$ be a holomorphic map between Riemann surfaces.

Then for any $p \in X$ there is a unique integer $m = m_{p,f} \geq 0$, such that there exist local parameters near p and $f(p)$ with f having the form $z \mapsto z^m$.

Proof of the proposition

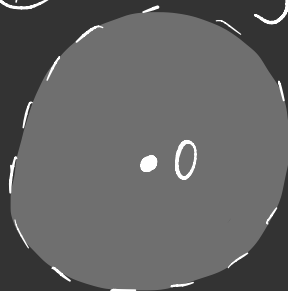
$$y = f(x)$$



$$\psi(y) = 0$$

$$\varphi(x) = 0$$

 $\mathbb{D}$

 Ω_x


$$g := \psi \circ f \circ \varphi^{-1}$$


 $\mathbb{D}$
 Ω_y


$$g(z) = \sum_{n=m}^{\infty} a_n z^n, \quad a_m \neq 0$$

$$w = \left(a_m z^m + a_{m+1} z^{m+1} + \dots \right)^{1/m}$$

$$z \mapsto w^m$$



Definition: The integer m given in the above proposition is the **multiplicity** of f at p , denoted $\text{mult}(f, p)$.

Definition: A point $p \in X$ is a **ramification point** of f if $\text{mult}(f, p) \geq 2$.

Exercise: The set of ramification points is discrete.

$\bullet p$

$\text{ramification points} \rightarrow X \xrightarrow{f} Y \leftarrow \text{branching points}$

Definition: Image of a ramification point is a **branching point**.

Theorem (Open mapping theorem)

Let $f: X \rightarrow Y$ be a holomorphic and non-constant map. Then f maps open subsets of X into open subsets of Y .

Theorem: Let X and Y be compact connected Riemann surfaces. Suppose that $f: X \rightarrow Y$ is a non-constant holomorphic map. Then f is surjective.

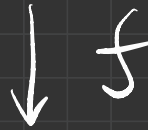
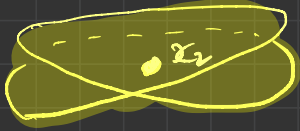
Degree of a holomorphic map

Proposition: Let $f : X \rightarrow Y$ be a non-constant holomorphic map between compact connected Riemann surfaces. For each $y \in Y$ we define

$$d_y(f) := \sum_{x \in \bar{f}^{-1}(y)} \text{mult}(f, x).$$

Then $d_y(f)$ is finite and constant (independent of y).

Proof of the proposition.



Degree of a holomorphic map.

Definition: Let $f: X \rightarrow Y$ be a non-constant holomorphic map between compact connected Riemann surfaces. The **degree of f** , denoted $\deg(f)$, is the integer $d_y(f)$ for arbitrary $y \in Y$.

Example.

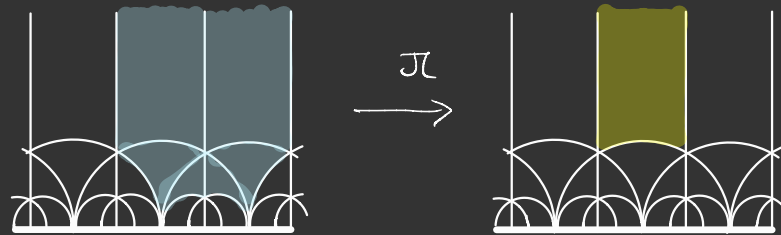
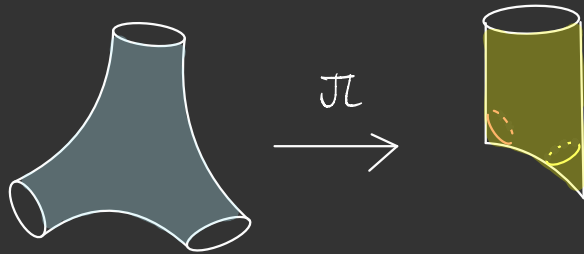
Let Γ be a congruence subgroup of Γ_1 .

Then the map

$$\pi: X(\Gamma) \rightarrow X(1)$$

has degree

$$\deg(\pi) = [\overline{\Gamma}_1 : \overline{\Gamma}]$$



Genus

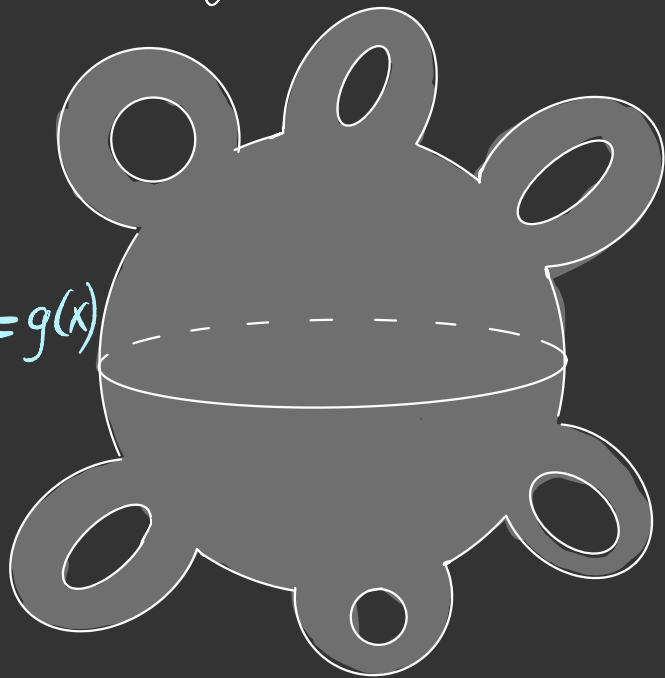
Classification of compact surfaces implies:

Topologically a compact Riemann surface X is a sphere with g handles.

Definition:

The number of handles is the **genus** of X . $g = g(X)$

Example: $g(X(\Gamma_1)) = 0$



Triangulation

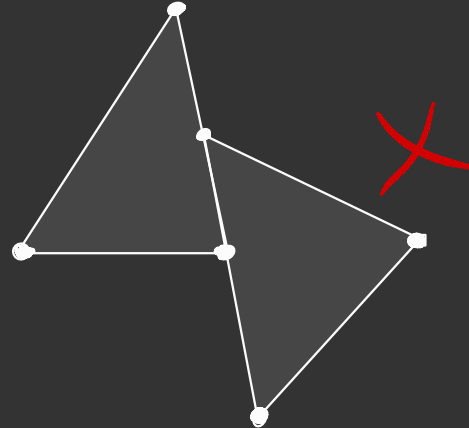
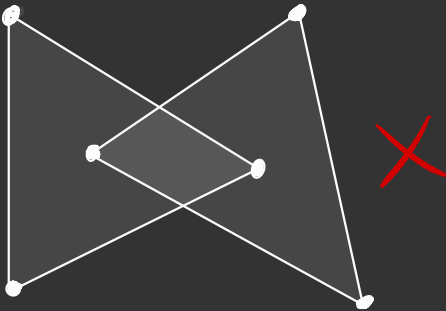
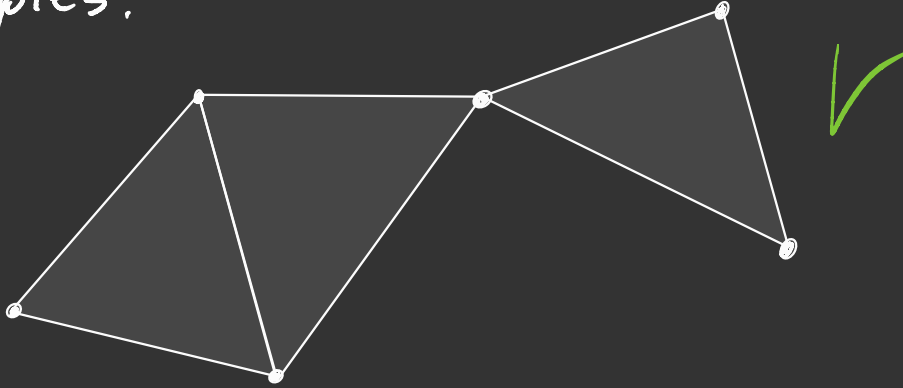
Definition: Let X be a topological space
(a surface in our case)

A triangulation of X is the collection
of maps

$$\tau_i : \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \begin{array}{c} T \\ \hline \end{array} \longrightarrow X \quad \text{Such that}$$

- $X = \bigcup_i \tau_i(T)$
- τ_i is a homeomorphism from T to $\tau_i(T)$
- if $\tau_i(T) \cap \tau_j(T)$ is non-empty then face
 $\tau_i(T) \cap \tau_j(T) = \tau_i(\text{edge/vertex}) = \tau_j(\text{edge/vertex})$
 $\tau_i|_{\text{face}} = \tau_j|_{\text{face}}$

Examples:



Euler's formula.

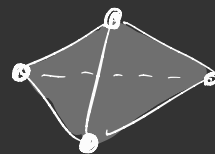
Theorem: Let X be a compact surface of genus g_X and let $\mathcal{T}$ be a **triangulation** of X .

Suppose that $\mathcal{T}$ consists of

t_2 triangles

t_1 edges

t_0 vertices



Then $t_2 - t_1 + t_0 = 2 - 2g_X$.

Riemann - Hurwitz formula

Theorem: Let X and Y be compact connected Riemann surfaces.

Let $f: X \rightarrow Y$ be a non-constant holomorphic map.

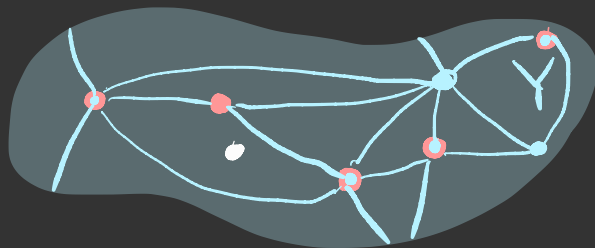
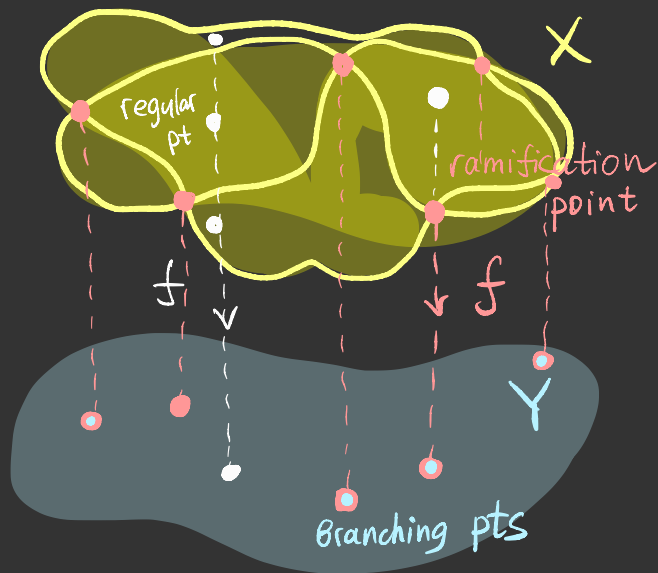
Then

$$2g_X - 2 = \deg(f) \cdot (2g_Y - 2) + \sum_{x \in X} (\text{mult}(f, x) - 1)$$

↑
we can only over
the ramification points.

Sketch of the proof.

Let T_Y be a triangulation of Y such that all branching points of f are in the set of vertices of T_Y



$$d = \deg(f)$$

Claim: Let $\tau: T \rightarrow Y$ be a triangle in a triangulation $\mathcal{T}_Y$. Then $\bar{f}^{-1}(\tau(T^\circ))$ is a disjoint union of d open triangles.

The maps $\bar{f}^{-1} \circ \tau: T \rightarrow X$ give a triangulation $\mathcal{T}_X$ of X .

Suppose that $\mathcal{T}_Y$ consists of t_0 vertices, t_1 edges, t_2 triangles then $\mathcal{T}_X$ consists of $d \cdot t_2$ triangles, $d \cdot t_1$ edges and t'_0 vertices

Let V_Y be the set of vertices of $\mathcal{T}_Y$ and V_X be the set of vertices of $\mathcal{T}_X$

Let V_Y be the set of vertices of T_Y and V_X be the set of vertices of T_X

$$\sum_{y \in V_Y} \sum_{\substack{x \in V_X \\ f(x)=y}} \text{mult}(f, x) = \sum_{y \in V_Y} d = \overset{\text{to}}{\#V_Y} \cdot d$$

$$\sum_{y \in V_Y} \sum_{\substack{x \in V_X \\ f(x)=y}} \text{mult}(f, x) = \sum_{x \in V_X} \text{mult}(f, x)$$

$$\sum_{x \in V_X} \text{mult}(f, x) - \#V_X = \sum_{x \in V_X} (\text{mult}(f, x) - 1) = \sum_{x \in X} (\text{mult}(f, x) - 1)$$

Thus $\#V_Y \cdot d - \#V_X = \sum_{x \in X} (\text{mult}(f, x) - 1).$

$\mathcal{T}_X$ consists of t'_0 vertices, t'_1 edges, t'_2 triangles

$$t'_0 - t'_1 + t'_2 = 2 - 2g_X$$

$\mathcal{T}_Y$ consists of t_0 vertices, t_1 edges, t_2 triangles

$$t_0 - t_1 + t_2 = 2 - 2g_Y$$

$$t'_1 = d \cdot t_1, \quad t'_2 = d \cdot t_2, \quad t'_0 = dt_0 - \sum_{x \in X} [\text{mult}(f, x) - 1]$$

$\Downarrow$

$$2 - 2g_X = d(2 - 2g_Y) - \sum_{x \in X} [\text{mult}(f, x) - 1]. \quad \square$$

Genus of a modular curve

Theorem: Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. Let $\pi: X(\Gamma) \rightarrow X(1)$ be the natural projection, and let $d = \deg(\pi)$. Let ε_2 and ε_3 denote the number of elliptic points of period 2 and 3 in $X(\Gamma)$, and ε_∞ the number of cusps of $X(\Gamma)$. Then the genus of $X(\Gamma)$ is

$$g = 1 + \frac{d}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}.$$

Proof.

We will use the Riemann-Hurwitz formula.

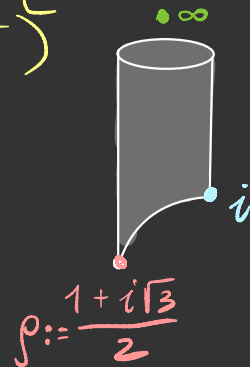
$$\pi : X(\Gamma) \rightarrow X(1)$$

$$2g(X(\Gamma)) - 2 = d(2g(X(1)) - 2) + \sum_{x \in X(\Gamma)} [\text{mult}(\pi, x) - 1]$$

We know that $g(X(1)) = 0$

It remains to compute

$$\sum_{x \in X(\Gamma)} [\text{mult}(\pi, x) - 1]$$



Consider the following cases:

- ⦿ point x is regular
- point x is elliptic on $X(\Gamma)$
- ⦿ point x is a cusp

Suppose that point $x \in X(\Gamma)$ is regular then:

→ $\pi(x)$ is not a cusp on $X(1)$

→ $\pi(x)$ can be regular $\pi': h^* \rightarrow X(1)$

elliptic of order 2 ($\pi'(i)$)
elliptic of order 3 ($\pi'(\rho)$)

mult $(\pi, x) = 1$ mult $(\pi, x) = 2$ mult $(\pi, x) = 3$

Suppose that $x \in X(\Gamma)$ is elliptic of order 2

then $\pi(x)$ is elliptic of order 2

$$\text{mult}(\pi, x) = 1$$

Suppose that $x \in X(\Gamma)$ is elliptic of order 3

then $\pi(x)$ is elliptic of order 3

$$\text{mult}(\pi, x) = 1$$

Suppose that $x \in X(\Gamma)$ is a cusp.

Then $\pi(x)$ is a cusp (~~the~~ cusp ∞ of $X(1)$)

$$\pi: \overline{\Gamma} \setminus \mathbb{P}^1(\mathbb{Q}) \rightarrow \overline{\Gamma}_1 \setminus \mathbb{P}^1(\mathbb{Q})$$

Let $\infty_1, \dots, \infty_n$ be the cusps of $X(\Gamma)$

Let Γ_{∞_i} be the stabilizer of ∞_i in $\overline{\Gamma}$

$$n_i := [\Gamma_{1, \infty_i} : \Gamma_{\infty_i}]$$

$$\text{mult}(\pi, \infty_i) = n_i$$

$$2g - 2 = d(2 \cdot 0 - 2) + \sum_{x \in X(\Gamma)} (\text{mult}(\pi, x) - 1)$$

x is a ramification point $\Rightarrow \pi(x) \in \{i, p, \infty\}$

$$\sum_{x \in \pi^{-1}(i)} (\text{mult}(\pi, x) - 1) = \sum_{\substack{\pi(x)=i \\ x \text{ is elliptic} \\ \text{of order 2}}} (1 - 1) + \sum_{\substack{\pi(x)=i \\ x \text{ is regular}}} (2 - 1)$$

$$= \# \pi^{-1}(i) - \varepsilon_2 = \frac{d}{2} - \frac{\varepsilon_2}{2}$$

$$d = \sum_{x \in \pi^{-1}(i)} \text{mult}(\pi, x) = (\# \pi^{-1}(i) - \varepsilon_2) \cdot 2 + \varepsilon_2$$

$$d = 2 \cdot \# \pi^{-1}(i) - \varepsilon_2 \Rightarrow \# \pi^{-1}(i) = \frac{d + \varepsilon_2}{2}$$

$$\sum_{x \in \bar{\pi}'(\rho)} (\text{mult}(\pi, x) - 1) = \sum_{\substack{x \in \bar{\pi}'(\rho) \\ x \text{ regular}}} (3-1) + \sum_{\substack{x \in \bar{\pi}'(\rho) \\ x \text{ elliptic} \\ \text{of order } 3}} (1-1)$$

$$= 2 \cdot (\# \bar{\pi}'(\rho) - \varepsilon_3) = \frac{2}{3}d - \frac{2}{3}\varepsilon_3$$

$$d = \sum_{x \in \bar{\pi}'(\rho)} \text{mult}(\pi, x) = 3(\# \bar{\pi}'(\rho) - \varepsilon_3) + \varepsilon_3$$

$$d = 3 \cdot \# \bar{\pi}'(\rho) - 2 \cdot \varepsilon_3 \Rightarrow \# \bar{\pi}'(\rho) = \frac{d}{3} + \frac{2}{3}\varepsilon_3$$

$$\sum_{x \in \bar{\pi}^{-1}(\infty)} (\text{mult}(\pi, x) - 1) = d - \# \bar{\pi}^{-1}(\infty) = d - \varepsilon_{\infty}$$

$$\sum_{x \in \bar{\pi}^{-1}(\infty)} (\text{mult}(\pi, x) - 1) = d - \varepsilon_{\infty}$$

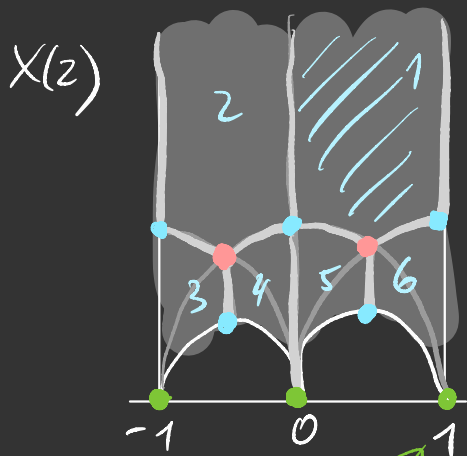
$$2g - 2 = d(2 \cdot 0 - 2) + \sum_{x \in \bar{\pi}^{-1}(i)} (\text{mult}(\pi, x) - 1) + \sum_{x \in \bar{\pi}^{-1}(p)} (\text{mult}(\pi, x) - 1) + \sum_{x \in \bar{\pi}^{-1}(\infty)} (\text{mult}(\pi, x) - 1)$$

$$= -2d + \frac{d}{2} - \frac{\varepsilon_2}{2} + \frac{2}{3}d - \frac{2}{3}\varepsilon_3 + d - \varepsilon_{\infty}$$

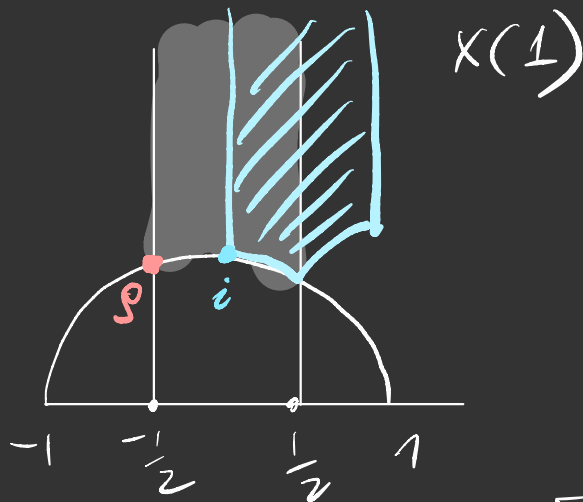
$$g = 1 + \frac{d}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_{\infty}}{2} \quad \square$$

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2} \right\}$$

∞



π



$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \Gamma(2)$$

$$\deg(\pi) = 6$$

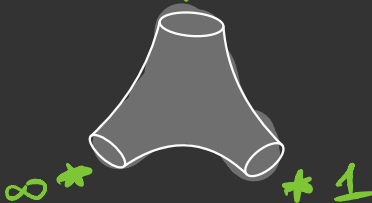
$$\varepsilon_3 = 0$$

$$\varepsilon_\infty = 3$$

$$\varepsilon_2 = 0$$

$$g = 1 + \frac{d}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}$$

$$g = 1 + \frac{6}{12} - \frac{0}{4} - \frac{0}{3} - \frac{3}{2} = 0$$



$$[\bar{\Gamma}_1 : \bar{\Gamma}(2)] = 6$$

$$\mathrm{Stab}_i(\bar{\Gamma}(2)) = \mathrm{id}$$

$$\mathrm{Stab}_p(\bar{\Gamma}(2)) = \mathrm{id}$$